International Journal of Theoretical Physics, Vol. 29, No. 10, 1990

KdV-Type Solitons in Multicomponent Relativistic Plasmas

B. Karmakar,[1] G. C. Das,[1] and Kh. Ibohanbi Singh[1,2]

Received May 3, 1990

A reductive perturbation technique is used to derive modified Korteweg–deVries (KdV) equations with different degrees of isothermality in a plasma, in order to study the existence and behavior of ion-acoustic solitary wave propagation ingoing in a multicomponent relativistic plasma. The solutions of the KdV equations are obtained. It is found that the presence of multiple ions and electrons in the relativistic plasma causes a different behavior regarding the formation of solitons in plasmas.

1. INTRODUCTION

Washimi and Taniuti (1966) first showed how solitary waves propagate in a simple plasma by deriving a nonlinear partial differential equation in the form of a Korteweg–deVries (KdV) equation. Later a number of authors (Taniuti and Wei, 1968; Su and Gardner, 1969; Schamel, 1973; Das and Tagare, 1975; Das, 1979; Tran and Hirt, 1974; Jones *et al.*, 1975; Goswami and Buti, 1976; Patraya and Chegeleshvilli, 1977; Abrol and Tagare, 1979) studied theoretically the existence and behavior of the solitons in multicomponent plasmas. Ikezi *et al.* (1970), Ikezi (1978), Nakamura (1982) and others have made significant contributions through their experimental investigations. Murthy *et al.* (1984) considered a plasma that includes multiple electrons and showed that the soliton behavior in the plasma exhibits fascinating results as compared to the plasma with multiple ions. Very recently, Das and Karmakar (1988, 1990) further analyzed the solitary waves in generalized multicomponent plasmas and showed that the propagation of solitary waves could be of interest for laboratory plasmas. But all the observations have been limited to a particular type of plasma and very

[1]Department of Mathematics, Manipur University, Imphal-795003, Manipur, India.
[2]Permanent address: Department of Mathematics, Modern College, Imphal-795001, India.

0020-7748/90/1000-1099$06.00/0

few results have been derived in relativistic plasmas. Das and Paul (1985) and Das *et al.* (1988) observed the propagation of solitary waves in relativistic isothermal plasmas and showed that the relativistic effect causes an ion-acoustic wave, accounting for a precursor due to which there is the possibility of breaking the soliton into many more solitons. They showed in a plasma with nonisothermal electrons that the presence of the relativistic effect does not exhibit such a peculiar phenomenon. However, the breaking up of the solitons could be avoided through the reduction of the nonisothermality of the plasma and we can derive the Kdv type soliton. Very recently Das *et al.* (1990) investigated the soliton behavior in a relativistic plasma with negative ions. They showed that the behavior of the solitons in a relativistic, weakly nonisothermal plasma in the presence of negative ions is different from the earlier study. They derived the modified Kdv equation and its solution and showed that the negative ions with relativistic effect modify the existence of the ion acoustic waves. Therefore, the possibility of getting compressional and rarefactive solitons in the plasma is significant.

Though the relativistic effect is small in the laboratory, if a high-power electromagnetic wave (power $\sim 10^{12-14}\,\mathrm{W/cm^2}$) is made to propagate through a plasma, the velocities of electrons and ions become relativistic. In nature, during solar bursts the velocity of the ejected plasma particles becomes weakly relativistic (~ 1000 km/sec). Therefore, in the present paper we are interested in the soliton behavior in relativistic plasmas that are isothermal, nonisothermal, and weakly nonisothermal. Our investigations show that the numerical results on the variation of the two-temperature electron ratio β and the relativistic effect ratio V_0/c especially exhibit new features of the solitons. By reducing the nonisothermal effects, we obtain a modified KdV equation and show that the soliton behavior differs from the earlier results.

2. MATHEMATICAL FORMULATION

We consider a collisionless unmagnetized relativistic plasma that includes two-temperature nonisothermal electrons. Following Das *et al.* (1990), we write the equation of continuity, the equation of motion for the adiabatic ions, and the Poisson equation for unidirectional propagation as

$$\frac{\partial \bar{n}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}}(\bar{n}\bar{v})=0 \tag{2.1}$$

$$\frac{\partial \bar{v}_\gamma}{\partial \bar{t}}+\bar{v}\frac{\partial \bar{v}_\gamma}{\partial \bar{x}}=-\frac{\partial \bar{\phi}}{\partial \bar{x}} \tag{2.2}$$

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2}=\bar{n}_{e1}+\bar{n}_{e2}-\bar{n} \tag{2.3}$$

where $\bar{n} = n_i/n_0$. The dimensionless parameters are defined as

$$\begin{aligned} \bar{\phi} &= e\phi(KT_e)^{-1}, & \bar{v}_\gamma &= \bar{v}(1-\bar{v}^2/\bar{c}^2)^{-1/2}, \\ \bar{c} &= c(KT_e/m_i)^{-1/2}, & \bar{v} &= v_i(KT_e/m_i)^{-1/2}, \\ \bar{x} &= x(KT_e/4\pi e^2 n_0)^{-1/2}, & \bar{t} &= t(4\pi e^2 n_0/m_i)^{1/2} \end{aligned} \tag{2.4}$$

where v_i is the velocity of the ions having the mass m_i and the number density n_i. Here $\eta_{e\alpha}$ and $T_{e\alpha}$ $(\alpha = 1, 2)$ stand for the number density and kinetic temperature of the electrons. n_0 is the ion-number density in the equilibrium state of the plasma. ϕ is the electrostatic potential and K is the Boltzmann constant.

Following Das *et al.* (1990), we introduce the nonisothermality of the plasma through the electron densities as

$$\begin{aligned} \bar{n}_{e1} &= \mu\left[1 + \frac{\bar{\phi}}{\mu+\nu\beta} - \frac{4}{3}b_1\left(\frac{\bar{\phi}}{\mu+\nu\beta}\right)^{3/2} + \frac{1}{2}\left(\frac{\bar{\phi}}{\mu+\nu\beta}\right)^2 + \cdots\right] \\ \bar{n}_{e2} &= \nu\left[1 + \frac{\beta\bar{\phi}}{\mu+\nu\beta} - \frac{4}{3}b_2\left(\frac{\beta\bar{\phi}}{\mu+\nu\beta}\right)^{3/2} + \frac{1}{2}\left(\frac{\beta\bar{\phi}}{\mu+\nu\beta}\right)^2 + \cdots\right] \end{aligned} \tag{2.5}$$

where $b_{1,2}$ are arbitrary constants depending on the electron temperatures through $\beta = T_{e1}/T_{e2}$.

Further, the following boundary conditions at $|x| \to \infty$ are assumed:

(i) $\bar{n}_{e1} \to \mu$ and $\bar{n}_{e2} \to \nu$, where μ and ν are the initial densities of the low- and high-temperature electron components.

(ii) $\bar{v} \to v_0$.

(iii) The overall charge neutrality condition is always maintained in the plasmas and is given by

$$\mu + \nu = 1 \tag{2.6}$$

2.1. Derivation of KdV Equation for Isothermal Electrons

In order to derive the KdV equation, we use the new stretched space-time variables ξ and τ given by the following relations:

$$\xi = \varepsilon^{1/2}(x - \lambda t), \qquad \tau = \varepsilon^{3/2} t \tag{2.7}$$

where ε is the expansion parameter and λ is the unknown phase velocity of the ion-acoustic wave to be obtained later.

First we consider the case of the isothermal plasma by putting $b_{1,2} = 0$ in the expressions (2.5). The plasma parameters (we omit the bars hereafter) are now expanded asymptotically as a power series in ε as

$$\begin{vmatrix} n \\ v \\ \phi \end{vmatrix} = \begin{vmatrix} 1 \\ v_0 \\ 0 \end{vmatrix} + \varepsilon \begin{vmatrix} n_1 \\ v_1 \\ \phi_1 \end{vmatrix} + \varepsilon^2 \begin{vmatrix} n_2 \\ v_2 \\ \phi_2 \end{vmatrix} + \cdots \tag{2.8}$$

Substituting (2.7) and (2.8) in the system of equations (2.1)-(2.3) and then collecting the lowest order terms in ε gives the following relations:

$$
\begin{aligned}
(\lambda - v_0)n_1 &= v_1 \\
(\lambda - v_0)(1+3v_0^2/2c^2)v_1 &= \phi_1 \\
n_1 &= \phi_1
\end{aligned}
\tag{2.9}
$$

from which the phase velocity λ of the wave is obtained as

$$\lambda = v_0 + (1+3v_0^2/2c^2)^{-1/2} \tag{2.10}$$

The same phase velocity λ is found as was obtained by Das *et al.* (1990), showing that the presence of multiple electrons does not have any effect on the phase velocity.

Now, the next higher order in ε gives the following relations:

$$(\lambda - v_0)\frac{\partial n_2}{\partial \xi} - \frac{\partial n_1}{\partial \tau} - \frac{\partial}{\partial \xi}(n_1 v_1) = \frac{\partial v_2}{\partial \xi} \tag{2.11}$$

$$
\begin{aligned}
&\left(1+\frac{3v_0^2}{2c^2}\right)\frac{\partial v_1}{\partial \xi} - 3(\lambda - v_0)\frac{v_0 v_1}{c^2}\frac{\partial v_1}{\partial \xi} - (\lambda - v_0)\left(1+\frac{3v_0^2}{2c^2}\right)\frac{\partial v_2}{\partial \xi} \\
&+\left(1+\frac{3v_0^2}{2c^2}\right)v_1\frac{\partial v_1}{\partial \xi} = -\frac{\partial \phi_2}{\partial \xi}
\end{aligned}
\tag{2.12}
$$

$$\frac{\partial^2 \phi_1}{\partial \xi^2} = \phi_2 - \frac{1}{2}\frac{(\mu + \nu\beta^2)}{(\mu + \nu\beta)^2}\phi_1^2 - n_2 \tag{2.13}$$

Eliminating n_2, v_2 from the relations (2.11)-(2.13) and using the first-order results (2.9)-(2.10), we derive the KdV equation in the following form:

$$\frac{\partial \phi_1}{\partial \tau} + A\phi_1\frac{\partial \phi_1}{\partial \xi} + B\frac{\partial^3 \phi_1}{\partial \xi^3} = 0 \tag{2.14}$$

where

$$A = \frac{\lambda - v_0}{2}\left[3 - \frac{\mu + \nu\beta^2}{(\mu + \nu\beta)^2} - \frac{3v_0}{c^2}(\lambda - v_0)^3\right]$$

$$B = \frac{\lambda - v_0}{2}$$

In order to get the solitary wave solution of equation (2.14), we introduce the new variable $\chi = \xi - U\tau$ with respect to a frame moving with the velocity U together with the following boundary conditions at $|\chi|\to\infty$:

(i) $\phi_1\to 0$.
(ii) $d\phi_1/d\chi\to 0$.
(iii) $d^2\phi_1/d\chi^2\to 0$.

Therefore, following Das *et al.* (1990), we obtain the stationary solution of the KdV equation (2.14) in the following form:

$$\phi_1=\phi_0\,\mathrm{Sech}^2(\chi/\delta_1) \tag{2.15}$$

where $\phi_0=(3U/A)$ is the amplitude and $\delta_1=2(B/U)^{1/2}$ is the width of the solitary wave.

2.2. Derivation of KdV Equation for Nonisothermal Electrons

Here we introduce the stretched coordinates ξ and τ given by the relations

$$\xi=\varepsilon^{1/4}(x-\lambda t),\qquad \tau=\varepsilon^{3/4}t \tag{2.16}$$

where ε and λ bear the same meaning as defined earlier.

In order to derive the case of the nonisothermal plasma, we have taken $b_{1,2}\neq 0$. As before, we now expand the plasma parameters asymptotically in a different form in powers of ε as

$$\begin{vmatrix} n \\ v \\ \phi \end{vmatrix} = \begin{vmatrix} 1 \\ v_0 \\ 0 \end{vmatrix} + \varepsilon \begin{vmatrix} n_1 \\ v_1 \\ \phi_1 \end{vmatrix} + \varepsilon^{3/2} \begin{vmatrix} n_2 \\ v_2 \\ \phi_2 \end{vmatrix} + \varepsilon^2 \begin{vmatrix} n_3 \\ v_3 \\ \phi_3 \end{vmatrix} + \cdots \tag{2.17}$$

Substituting the relations (2.16) and (2.17) in the basic equations (2.1)-(2.3) and comparing the lowest order terms in ε, we obtain the same phase velocity λ, without any changes as compared to the earlier expression obtained in (2.10).

Now, the next higher order in ε yields the equations

$$(\lambda-v_0)\frac{\partial n_2}{\partial\xi}-\frac{\partial n_1}{\partial\tau}=\frac{\partial v_2}{\partial\xi} \tag{2.18}$$

$$(\lambda-v_0)\frac{\partial v_2}{\partial\xi}-\frac{\partial v_1}{\partial\tau}=\left(1+\frac{3v_0^2}{2c^2}\right)^{-1}\frac{\partial\phi_2}{\partial\xi} \tag{2.19}$$

$$\frac{\partial^2\phi_1}{\partial\xi^2}=\phi_2-\frac{4}{3}\left(\frac{b_1\mu+b_2\nu\beta^{3/2}}{(\mu+\nu\beta)^{3/2}}\right)[\phi_1]^{3/2}-n_2 \tag{2.20}$$

Following the usual procedure and after a mathematical manipulation, we obtain a single nonlinear differential equation in ϕ_1 in the form of the following KdV equation:

$$\frac{\partial\phi_1}{\partial\tau}+C[\phi_1]^{1/2}\frac{\partial\phi_1}{\partial\xi}+B\frac{\partial^3\phi_1}{\partial\xi^3}=0 \tag{2.21}$$

where

$$C=\frac{(\lambda-v_0)(b_1\mu+b_2\nu\beta^{3/2})}{(\mu+\nu\beta)^{3/2}}$$

and B was defined earlier.

The corresponding solution of the KdV equation (2.21) is obtained as

$$\phi_1=\phi_0\,\mathrm{Sech}^4(\chi/\delta_2) \tag{2.22}$$

where $\phi_0=225U^2/64C^2$ is the amplitude and $\delta_2=4(B/U)^{1/2}$ is the width of the ion-acoustic solitary wave.

We now analyze the existence and behavior of the KdV type solitons. The characteristic variation of soliton features is shown numerically in the figures. The amplitude variation with the electron-temperatures ratio is shown in Figure 1 and is compared with the result for simple plasmas. The dotted lines represent the characteristics of the solitary waves in a nonrelativistic multiple electron plasma. Figure 1 shows that the behavior of the ion-acoustic waves for the relativistic and nonrelativistic multiple electron plasmas in the isothermal case are of the same nature and shows little decrease in the amplitude variation. Similar characteristics is also observed in the case of a nonisothermal plasma. Figure 2 shows the variation of the amplitude ϕ_0 with the relativistic effect v_0/c in the isothermal and nonisothermal plasmas. This observation indicates the identical behavior of the solitary waves at $\beta=0.2$ and $\beta=0.5$. The figure shows that the amplitude ϕ_0 of the solitary waves increases with increasing values of v_0/c, indicating that the amplitude could be very large at a higher value of v_0/c and as such the solitons will break down and the formation of the solitary waves will not be possible. At $\beta=0.5$, the amplitude of the solitary wave is less than that obtained at $\beta=0.2$. This implies that the existence of the solitary waves is possible for higher values of v_0/c and β. Thus, the present observation indicates that the presence of the multiple electron temperature has an important role in the existence of the ion-acoustic waves.

The variation of the potential ϕ_1 of the solitary wave with the widths δ_1 and δ_2 is shown in Figures 3 and 4, and we see the same nature in both the isothermal and nonisothermal plasmas. Here the potential of the solitary wave decreases with the increasing values of the width δ_1 of the solitary

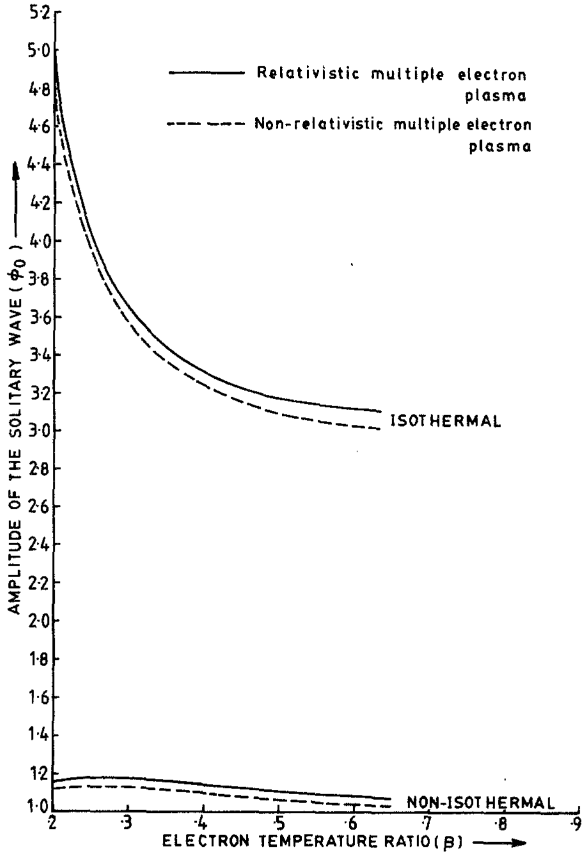

Fig. 1.

wave. In the figures, the dotted lines indicate the behavior of the potential with the widths of the solitons in a simple relativistic plasma. In the present study the potential ϕ_1 decreases along a curved path, while the earlier study showed that the path is a straight line. Figures 3 and 4 show that the potential is higher at $\beta = 0.2$ than that for $\beta = 0.5$. The present study points that the value of the potential decreases for higher value of β.

Since we know from our earlier results that a weak nonisothermality leads to a different behavior of the solitons in plasmas, we consider the case of weak nonisothermality. In this case the parameters b_1 and b_2 arising due to the nonisothermality of the plasma are assumed to be small and we consider $b_{1,2} = \varepsilon^{1/2} b_{1,2}$, where $b_{1,2} > 0$. The scheme of the perturbation expansion of the field variables and stretching coordinates will be different as compared to the case of nonisothermality. Here we consider the stretched

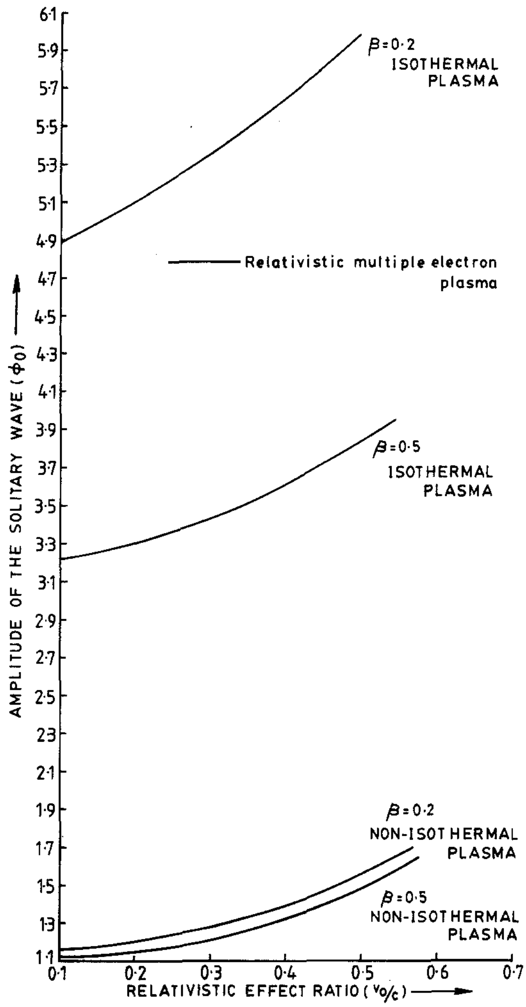

Fig. 2.

coordinates and the expansion of the parameters as in the case of isothermal plasmas, as given in expressions (2.7) and (2.8).

Following a similar procedure, we first get the phase velocity, which is the same as that defined in (2.10), and finally, the next higher order in ε gives the modified KdV equation as

$$\frac{\partial \phi_1}{\partial \tau}+[A\phi_1+C(\phi^{(1)})^{1/2}]\frac{\partial \phi_1}{\partial \xi}+B\frac{\partial^3 \phi_1}{\partial \xi^3}=0 \tag{2.23}$$

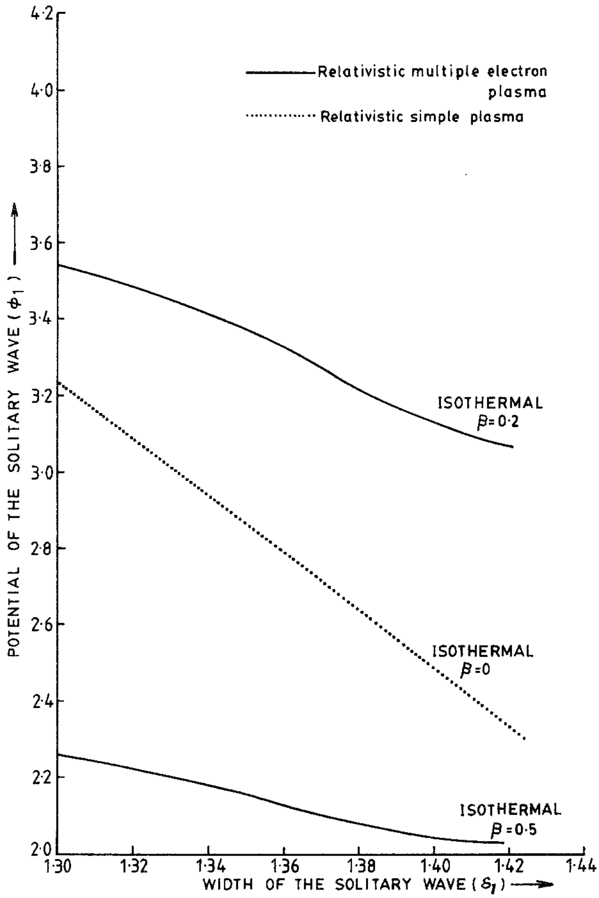

Fig. 3.

where

$$A=\frac{\lambda-v_0}{2}\left[3-\frac{\mu+\nu\beta^2}{(\mu+\nu\beta)^2}-\frac{3v_0}{c^2}(\lambda-v_0)^3\right]$$

B and C are similar to those defined earlier.

The corresponding solitary wave solution is obtained in the following form:

$$\phi_1=\left[\frac{4C}{15U}+\left(\frac{16C^2}{225U^2}+\frac{A}{3U}\right)^{1/2}\cosh\left(\frac{\chi}{\delta_3}\right)\right]^{-2} \tag{2.24}$$

where $\delta_3=2\ (B/U)^{1/2}$ is the width of the solitary wave.

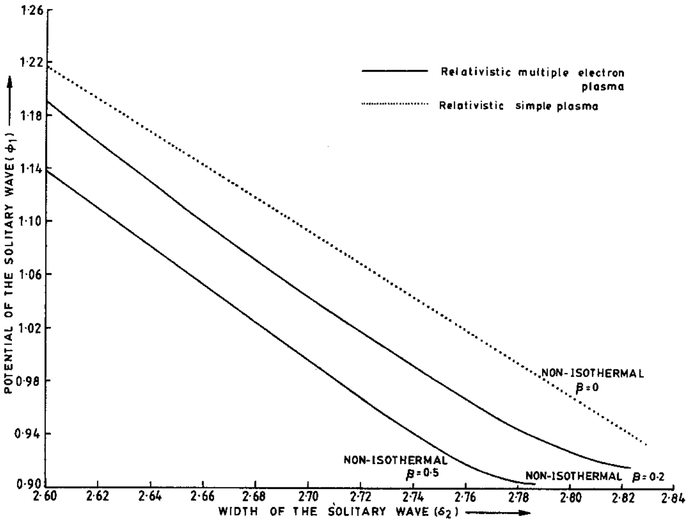

Fig. 4.

Here the basic difference from the earlier KdV-type soliton solution is that the solution depends on the values of C and A. Multiple electrons contribute to the formation of the ion-acoustic solitary waves. From expression (2.24) we have an amplitude ϕ_1 which is quite different from the result derived in Das *et al.* (1990). But the width δ_3 is unchanged and this plays a significant part in the formation of solitons with various amplitude variations arising in isolation with the plasma parameters. Figure 5 shows the variation of the potential with the width of the solitary waves in a weakly nonisothermal plasma and the dotted line indicates the variation of the potential without the effect of the multiple electron temperatures ratio.

The plot in Figure 5 shows that the potential ϕ_1 decreases with increasing value of the width δ_3 of the ion-acoustic waves, whereas in the earlier work (Das *et al.*, 1990), the potential ϕ_1 decreases, showing a drastic change in the character of the solitary waves in a multiple temperature electron plasma.

3. CONCLUSIONS

In the present study we have investigated ion-acoustic waves, taking account of the combined effects of multiple electron temperatures and the relativistic effect. After deriving the KdV equations for isothermal, non-isothermal, and weakly nonisothermal electron plasmas, we were able to show that the solutions for the KdV-type solitons are quite different from

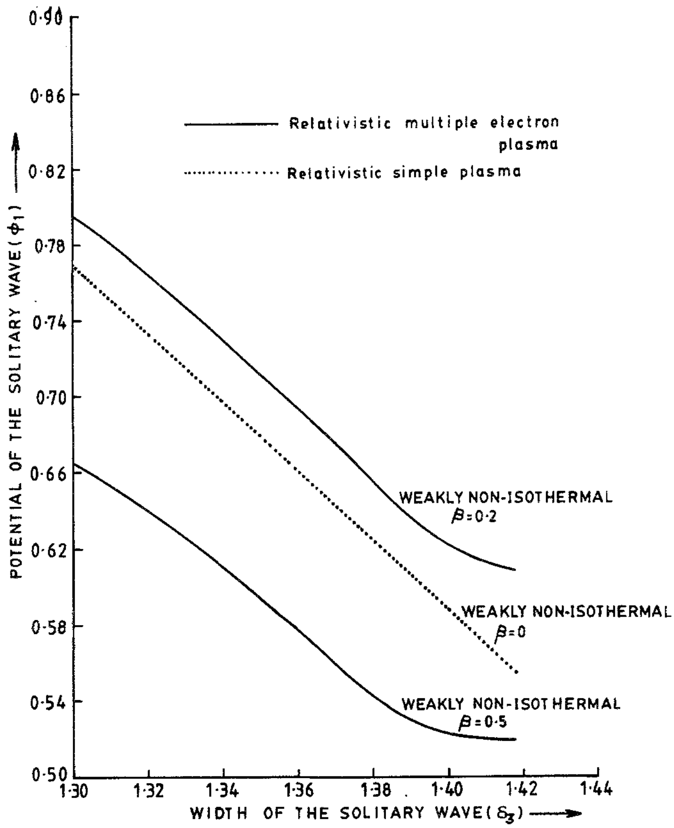

Fig. 5.

the results obtained by Das *et al.* (1990). Different electron temperatures play a dominating role for the formation of the solitary waves. The numerical results show that the range of the multiple electron temperature ratio β is limited and the amplitude of the ion-acoustic waves depends on the effects of β and v_0/c. Thus, we conclude that solitons with isothermal as well as the nonisothermal multiple electron temperatures in the presence of the relativistic effect might be observed experimentally, revealing fascinating soliton behavior in a plasma. One has to be careful about the choice of the range of the two electron-temperature ratio β as well as the values of v_0/c in order to observe the prominent characteristics of the solitons in a plasma.

REFERENCES

Abrol, P. S., and Tagare, S. G. (1979). *Physics Letters A*, **75**, 74.
Das, G. C. (1979). *Plasma Physics*, **21**, 257.

Das, G. C., and Karmakar, B. (1988). *Canadian Journal of Physics*, **66**, 79.
Das, G. C., and Karmakar, B. (1990). *Australian Journal of Physics*, **43**, 1
Das, G. C., and Paul, S. N. (1985). *Physics of Fluids*, **28**, 823.
Das, G. C., and Tagare, S. G. (1975). *Plasma Physics*, **17**, 1025.
Das, G. C., Karmakar, B., and Paul, S. N. (1988). *IEEE Transactions on Plasma Science*, **16**, 22.
Das, G. C., Karmakar, B., and Ibohanbi Singh, K. (1990). *IEEE Transactions on Plasma Science*, **18**, 86.
Goswami, B. N., and Buti, B. (1976). *Physics Letters A*, **57**, 149.
Ikezi, H. (1978). In *Solitons in Action*, K. E. Lonngren and A. Scott, eds.
Ikezi, H., Taylor, R. J., and Baker, D. R. (1970). *Physical Review Letters*, **25**, 11.
Jones, W. D., Lee, A., Gleeman, S. M., and Doucet, H. J. (1975). *Physical Review Letters*, **35**, 1349.
Murthy, P. N., Tagare, S. G., and Abrol, P. S. (1984). *Canadian Journal of Physics*, **62**, 45.
Nakamura, Y. (1982). *IEEE Transactions on Plasma Science*, **PS-10**(3), 180.
Patraya, A. D., and Chegeleshvilli, G. D. (1977). *Soviet Journal of Plasma Physics*, **3**, 736.
Schamel, H. (1973). *Journal of Plasma Physics*, **9**, 377.
Su, C. H., and Gardner, C. S. (1969). *Journal of Mathematical Physics*, **10**, 536.
Taniuti, T., and Wei, C. C. (1968). *Journal of the Physical Society of Japan*, **24**, 941.
Tran, M. Q., and Hirt, P. J. (1974). *Plasma Physics*, **16**, 617.
Washimi, H., and Taniuti, T. (1966). *Physical Review Letters*, **17**, 996.